Geometric Graphs

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Indian Institute of Science, Bangalore

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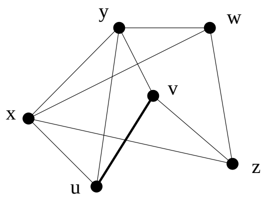
Geometric Graph



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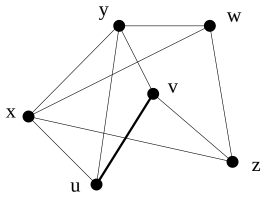

- $V =$ set of geometric objects

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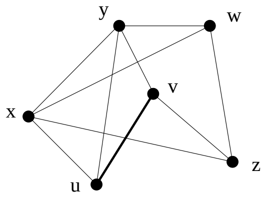

- $V =$ set of geometric objects
- $E = \{(u, v)\}$ based on some geometric condition (ex. intersection)

Questions on Geometric Graphs

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- Computational questions
  - Efficient Algorithm
  - Approximation

Geometric graphs

- V - set of geometric objects
- E - object i and j satisfy certain geometric condition

- Broad classes of geometric graphs (based on edge condition)

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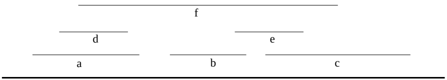
  - Distance based graphs

Intersection Graphs

- Interval Graph - Classic example
- S - set of geometric objects s_i (intervals on the real line)



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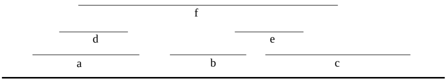

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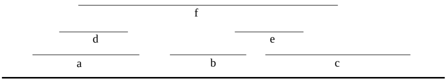
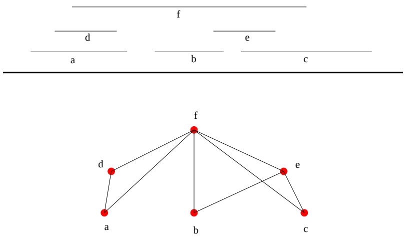


- V - set of objects s_i
- $(s_i, s_j) \in E$ if objects s_i and s_j intersect

- S - set of intervals on the line



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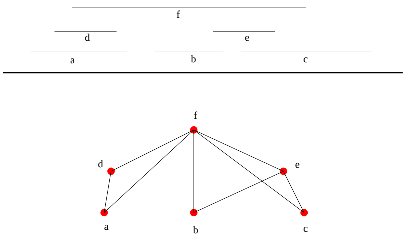

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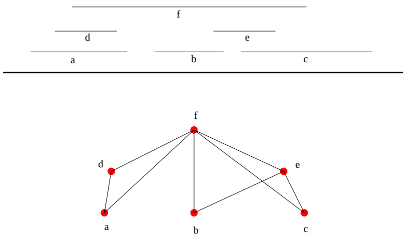

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  - Jobs: $(6, 12), (8, 10), (7, 13), (9, 17), (11, 15), (12, 16), (15, 18)$

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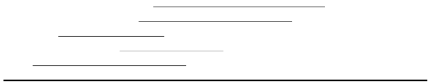

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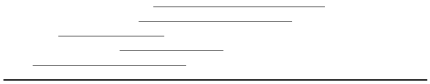

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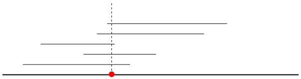
  - Construct a point p that is contained in all the intervals



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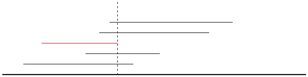

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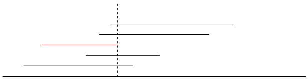

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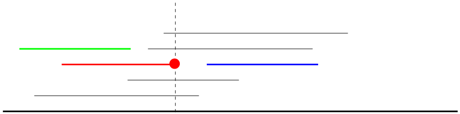
- Proof by contradiction



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Let C be a collection of convex objects in R^d. If every $d+1$ objects in C have a common intersection, then all the objects in C have a common intersection.

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- We proved this theorem for $d = 1$
- Proof using different approaches
  - Radon's theorem
  - Induction
  - Shrinking ball technique
  - Brouwer's theorem
  - **Constructive/Extremal proof**

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    with **Proof of correctness**

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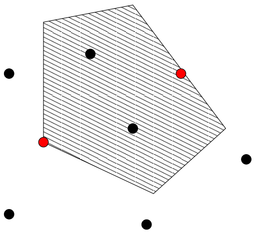
Proximity Graphs

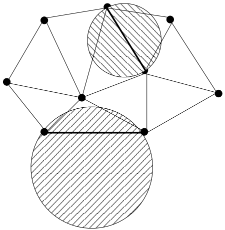
- P - point set in plane
- $R_{i,j}$ - proximity region defined by i and j



- V - point set P
- $(i,j) \in E$ if $R_{i,j}$ is empty
- Examples - Delaunay, Gabriel, Relative Neighborhood Graph
- Applications - Graphics, wireless networks, GIS, computer vision

Delaunay Graph - Classic Example

- P - point set in plane



- V - point set P
- $(i, j) \in E$ if $\exists$ some empty circle thro' i and j
- Triangle (i, j, k) if circumcircle(i, j, k) is empty (Equivalent condition)
- Applications - Graphics, mesh generation, computer vision, etc.

Questions on Delaunay Graph

- Combinatorial - Bounds on
  - Maximum size of edge set?
  - Chromatic number?
  - Maximum independent set?

  (Over all possible point sets P)
- Computational
  - Efficient Algorithm

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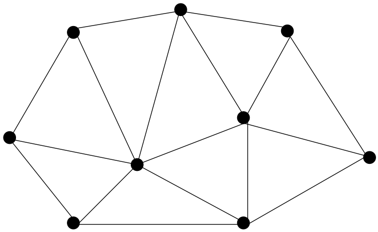

- Observations:

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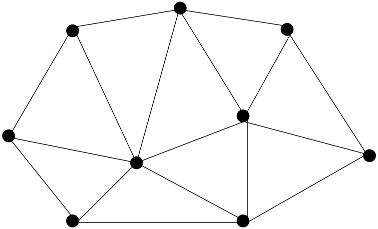

- Observations: **Planar?**

Delaunay Graph - Planar

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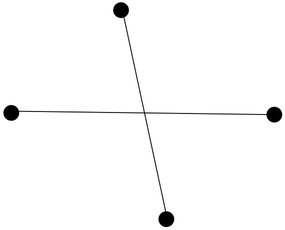

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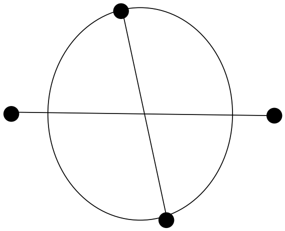

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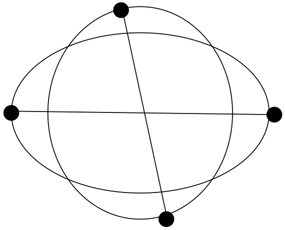

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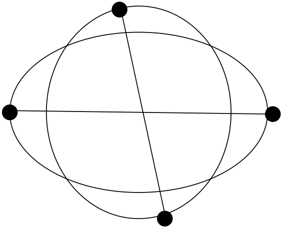

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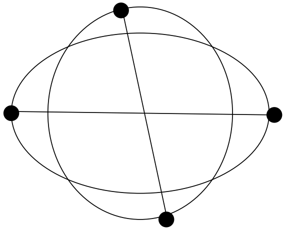

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- One endpoint of an edge lies within the other circle
  - Contradiction
- Alternate proof using angles

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  - ≤ 4 (Four color theorem)
- Maximum independent set

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  - $\geq n/4$ (Chromatic number)

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