Introduction to Probability

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$$\frac{13}{52} = \frac{1}{4} \tag{3}$$

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$$\frac{\text{number of favourable outcomes}}{\text{number of all possible outcomes}} \tag{4}$$

Some Terminology

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Sample Space **S**={{**HH**},{**HT**},{**TH**},{**TT**}}

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$$\mathbf{E} = \Big\{\{\{\mathbf{HH}\}\}, \{\{\mathbf{HT}\}\}, \{\{\mathbf{TH}\}\}, \{\{\mathbf{TT}\}\}, \{\{\mathbf{HH}\},\{\mathbf{HT}\}\}, \{\{\mathbf{HH}\},\{\mathbf{TH}\}\}, \{\{\mathbf{HH}\},\{\mathbf{TT}\}\}, \{\{\mathbf{HT}\},\{\mathbf{TH}\}\}, \{\{\mathbf{HT}\},\{\mathbf{TT}\}\}, \{\{\mathbf{TH}\},\{\mathbf{TT}\}\}, \{\{\mathbf{HH}\},\{\mathbf{HT}\},\{\mathbf{TH}\}\}, \{\{\mathbf{HH}\},\{\mathbf{HT}\},\{\mathbf{TT}\}\}, \{\{\mathbf{HH}\},\{\mathbf{TH}\},\{\mathbf{TT}\}\}, \{\{\mathbf{HT}\},\{\mathbf{TH}\},\{\mathbf{TT}\}\}, \{\{\mathbf{HH}\},\{\mathbf{HT}\},\{\mathbf{TH}\},\{\mathbf{TT}\}\}, \{\Phi\}\Big\}$$

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- Probability of *Event* of interest

The probability assignment $\mathbf{P} : \mathbf{E} \rightarrow [0, 1]$

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Certain Properties

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- $\mathbf{P}(\Phi) = 0$.

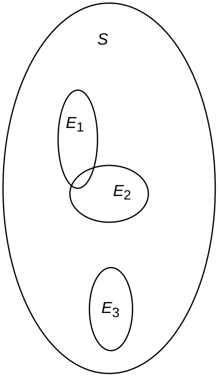


Figure: Abstract Probability Space

- World is not as simple as Coins, Dice and Cards.

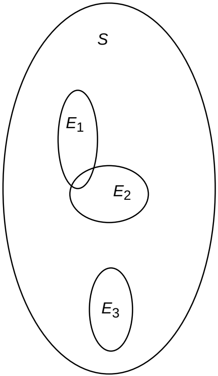
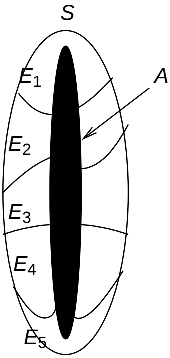
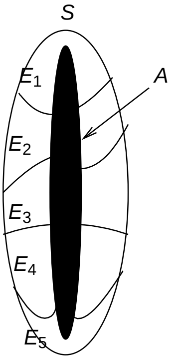


Figure: Abstract Probability Space

- World is not as simple as Coins, Dice and Cards.
- Any abstraction is just as useful as $(a + b)^2 = a^2 + b^2 + 2ab$.



Here we are interested in probabilities of events given an event A has occured.



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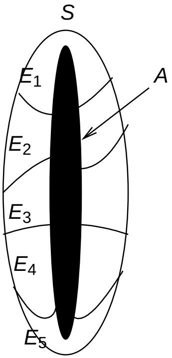
$$\mathbf{P}(E_1|A) = \frac{\mathbf{P}(E_1 \cap A)}{\mathbf{P}(A)} \tag{6}$$



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$$\mathbf{P}(\{\{\mathbf{TT}\},\{\mathbf{HT}\},\{\mathbf{TH}\}\}) = p \times p + (1-p) \times p + p \times (1-p) \tag{8}$$

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Examples of independent events

- Getting ‘Tail’ in the first toss is independent of getting ‘Head’ in the second toss.
- When I toss a coin and roll a die simultaneously the outcomes are independent of each other.
- How many glasses of water I drink is independent of whether it will rain Tomorrow.

The above examples are **Correct** but **Useless**

$E_1, E_2, \ldots, E_n$ be disjoint sets such that $\bigcup_i E_i = \mathbf{S}$, let A be any event

$$\mathbf{P}(A) = \mathbf{P}(A|E_1) \times \mathbf{P}(E_1) + \mathbf{P}(A|E_2) \times \mathbf{P}(E_2) + \ldots + \mathbf{P}(A|E_n) \times \mathbf{P}(E_n) \tag{9}$$

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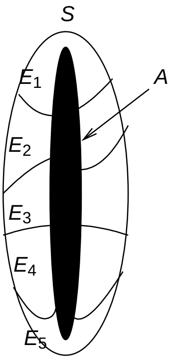
This is similar to the *cut-off* mark calulations.

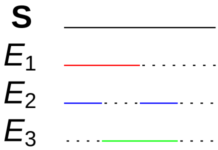


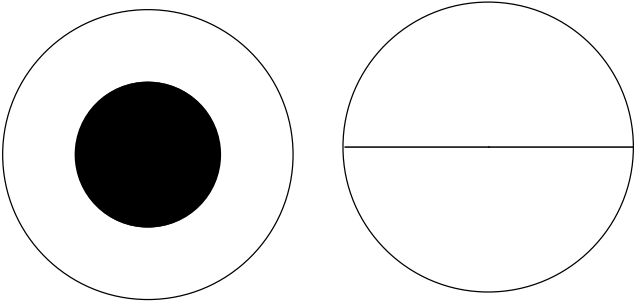
This leads to the following relation

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(B|A) \times \mathbf{P}(A)}{\mathbf{P}(B|A) \times \mathbf{P}(A) + \mathbf{P}(B|A^c) \times \mathbf{P}(A^c)}. \tag{10}$$

Independent Events Contd

$$\mathbf{S} = [0,1], E_1 = [0, \frac{1}{2}], E_2 = [0, \frac{1}{4}] \cup [\frac{1}{2}, \frac{3}{4}], E_3 = [\frac{1}{4}, \frac{3}{4}]. \quad (11)$$





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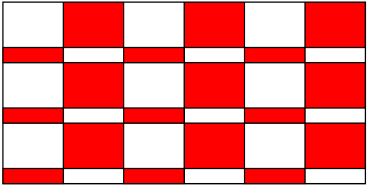

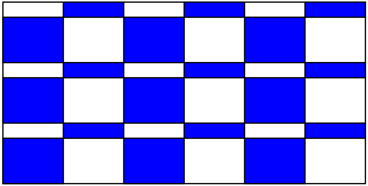

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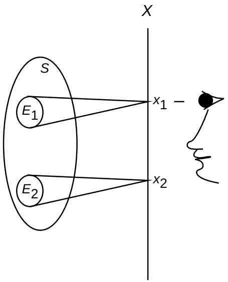
What is it?
It is a function $X : \mathbf{S} \to \mathbf{R}$.



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$$\begin{aligned}
\mathbf{S} &= \{1, 2, 3, 4, 5, 6\} \\
X(1) &= 1 \\
X(2) &= 2 \\
X(3) &= 1 \\
X(4) &= 2 \\
X(5) &= 1 \\
X(6) &= 2
\end{aligned} \tag{12}$$

One can define another R.V. Y where Rs.$x/2$ when x shows up.

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$$
\begin{aligned}
\mathbf{S} &= \{1, 2, 3, 4, 5, 6\} \\
Y(1) &= \frac{1}{2} \\
Y(2) &= 1 \\
Y(3) &= \frac{3}{2} \\
Y(4) &= 2 \\
Y(5) &= \frac{5}{2} \\
Y(6) &= 3
\end{aligned} \tag{13}
$$

$\mathbf{P}(X = 2)$, $\mathbf{P}(X = 1, Y = 1)$, $\mathbf{P}(X = 1, Y = -1)$.

Probability Mass Function

Associated with a R.V. X we are interested in its *probability mass function* (pmf)

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Let us compute the pmfs for X and Y.

$$f_X(1) = \mathbf{P}(X = 1) = \frac{1}{2}, f_X(2) = \mathbf{P}(X = 2) = \frac{1}{2} \tag{14}$$

$$\begin{aligned}
f_Y(1) = \mathbf{P}(Y = 1) &= \frac{1}{2}, & f_Y(2) = \mathbf{P}(Y = 2) = && \frac{1}{2} \\
f_Y(3) = \mathbf{P}(Y = 3) &= \frac{1}{2}, & f_Y(4) = \mathbf{P}(Y = 4) = && 0 \\
f_Y(5) = \mathbf{P}(Y = 5) &= 0, & f_Y(6) = \mathbf{P}(Y = 6) = && 0 \\
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\end{aligned} \tag{15}$$

$f(x)$ can also be thought of as the frequency of occurence of x.

Cumulative Distribution Function

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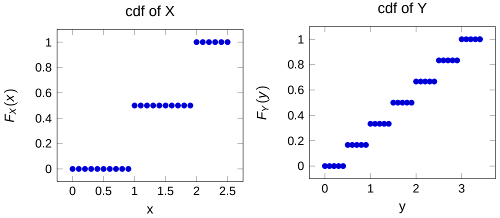
This is the *cumulative distribution function*



Bernoulli Random Variable X is either *Success*-1 or *Failure*-0. Thus X takes values 1 or 0.

$$\mathbf{P}(X = 0) = 1 - p,$$
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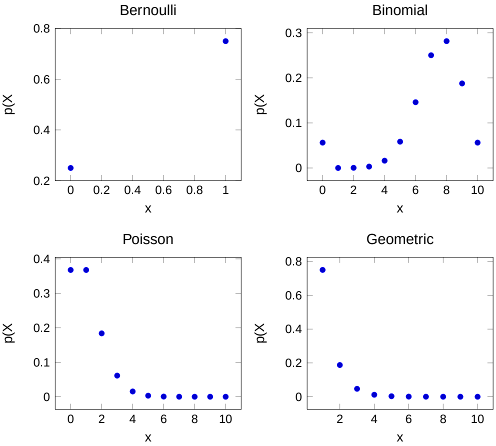
Poisson Random Variable X takes values $0, 1, 2, 3, \ldots$ with

$$\mathbf{P}(X = k) = \exp(-\lambda) \frac{(\lambda)^k}{k!} \tag{18}$$

Useful to model number of arrivals in unit time.

Useful Random Variables



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- However the cdf still makes sense.
- Equivalent to the pmf we define the *probability density function* (pdf).

- Till now the Random Variables assumed only discrete values (finite or countable).
- Consider a Random Variable that takes a *continuum* of values.
- Then $\mathbf{P}(X = x)$ is 0 for all x and pmf does not make sense.
- However the cdf still makes sense.
- Equivalent to the pmf we define the *probability density function* (pdf).
- The pdf is derivative of the cdf.

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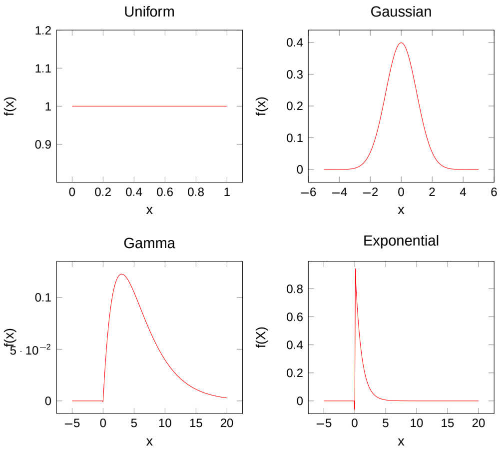
Useful Continuous Random Variables



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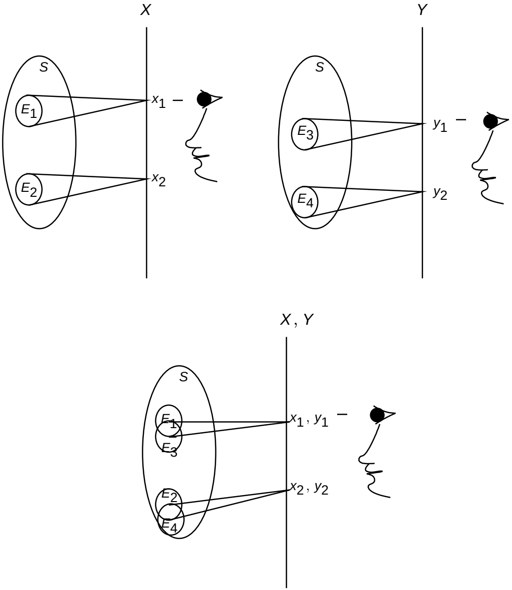
- $Var(X)$ is always positive.
- Verify that $Var(X) = \mathbf{E}(X^2) - (\mathbf{E}(X))^2$.

Two Random Variables



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We also can talk about the *conditional* $f_{X|Y=y}(x)$, distribution of X given that Y assumes a value y.

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F_{XY}(x,y) &= \mathbf{P}(X \le x, Y \le y) \\
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- The pdf (also the pmf) $f_{XY}(x,y) = f_X(x)f_Y(y)$.

In case of independent R.Vs

- We can determine the joints given the marginals.
- The conditional distribution is same as the marginal.

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$$\begin{aligned}\mathbf{P}(X = 1) = 0.5, \mathbf{P}(Y = 1) = 0.5 \\ \mathbf{P}(X = 0) = 0.5, \mathbf{P}(Y = 0) = 0.5\end{aligned} \tag{24}$$

Illustration with Dice

X-outcome of first toss, Y-outcome of second toss.

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$X \backslash Y$					
(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Illustration with Dice

Z-sum of the two outcomes.

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

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Conditional Mean and Covariance

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$$\mathbf{E}\left[(X - \mathbf{E}(X))(Y - \mathbf{E}(Y))\right] \quad (25)$$

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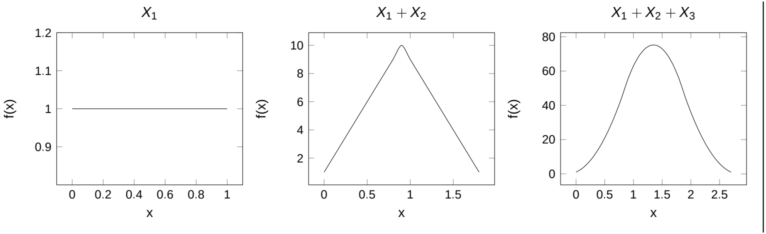

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- If $Y = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$. Then $f_Y is \mathcal{N}(0,1)$, i.e., Gaussian R.V with mean 0 and variance 1.