Randomization at Work: An Introduction to Randomized Algorithms

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- Divide-and-conquer algorithm.
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3. **Combine:** The array will be trivially sorted using the above steps.

QuickSort : Partitioning

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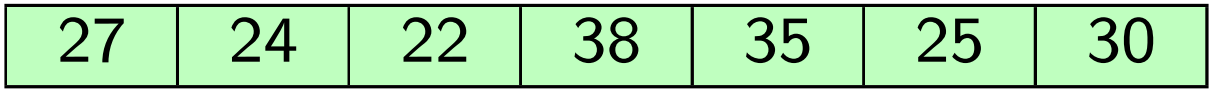

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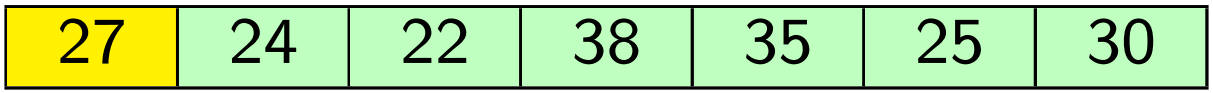

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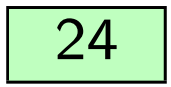

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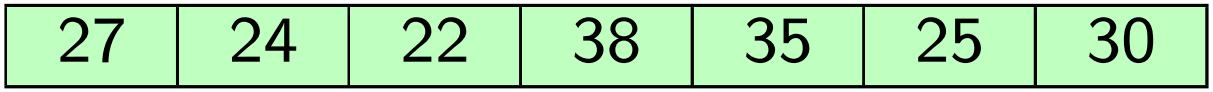

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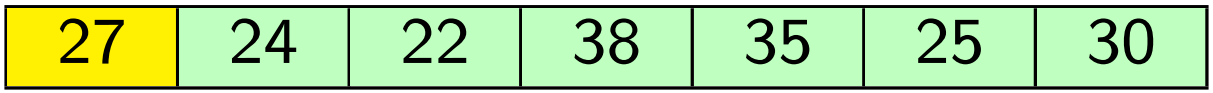

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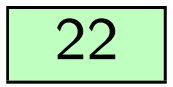

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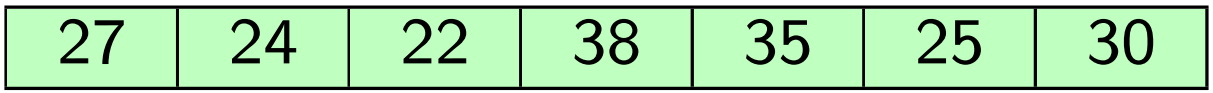

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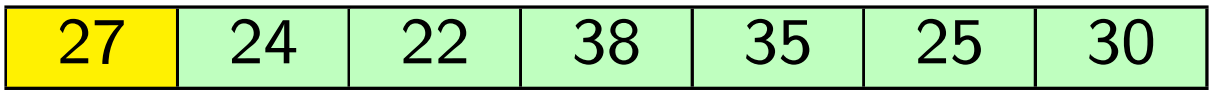
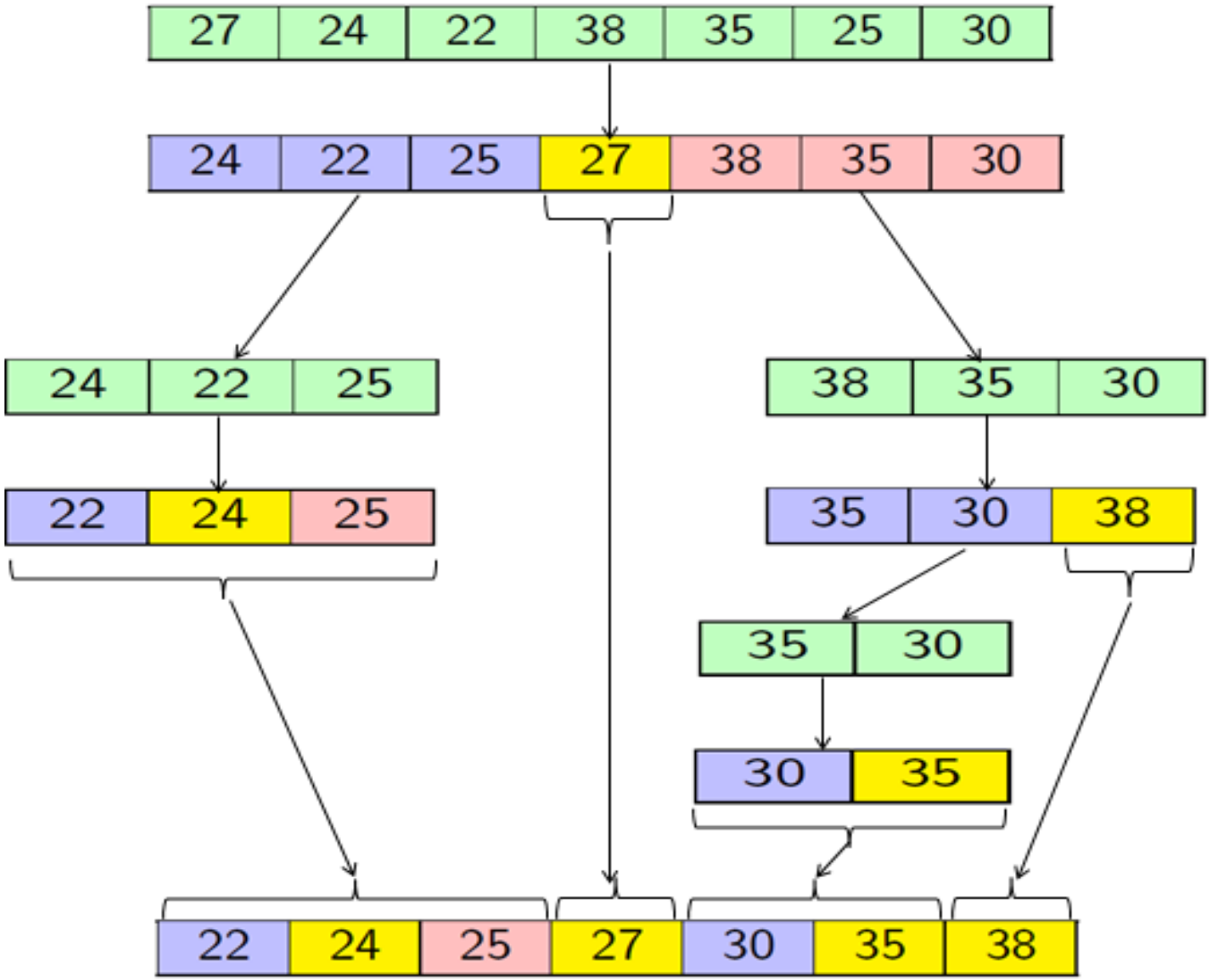


2. Partition

Final Partitioned Array

QuickSort: Recursion



Worst Case Analysis of QuickSort

- Worst Case occurs when you always partition around the minimum or maximum element.
- One side of the partition always has $n-1$ elements and the other has no elements.

$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= T(n-2) + \Theta(n-1) + \Theta(n) \\
&\cdots \\
&= \Theta(1) + \Theta(2) + \cdots + \Theta(n-1) + \Theta(n) \\
&= \Theta(n^2)
\end{aligned}
$$

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- Pick the pivot randomly to partition each time.
- Therefore, we almost never allow worst case to occur, whatever the type of the input array.
- In adversarial environment, we confuse the adversary with the random choice of pivot.

Analysis of Randomized QuickSort

Let X be the total number of comparisons. Let $[x_1, x_2, \cdots, x_n]$ be the input array, and $[y_1, y_2, \cdots, y_n]$ be the output sorted array.

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Define : X_{ij} to be a indicator random variable such that

$$X_{ij} = \begin{cases} 1 & \text{if } y_i \text{ and } y_j \text{ are compared} \\ 0 & \text{otherwise} \end{cases}$$

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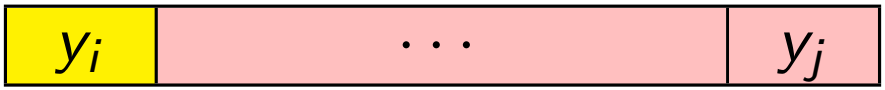
Case 1: y_i was chosen as the pivot. y_i and y_j are compared in this case.



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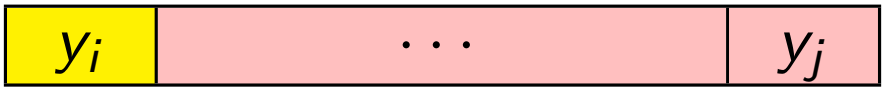
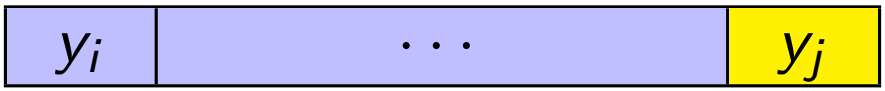


Case 2: Some element from y_{i+1} to y_{j-1} was chosen as the pivot. y_i and y_j are not compared in this case.

Analysis of Randomized QuickSort

Case 3: y_j was chosen as the pivot. y_i and y_j are compared in this case.



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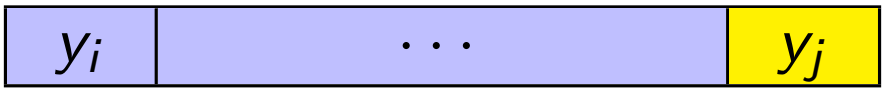

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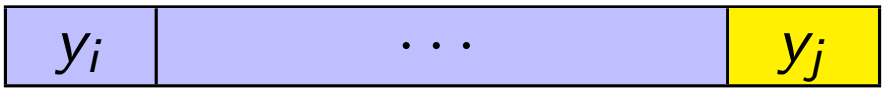

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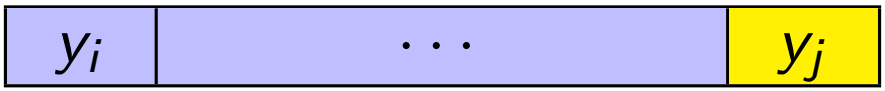

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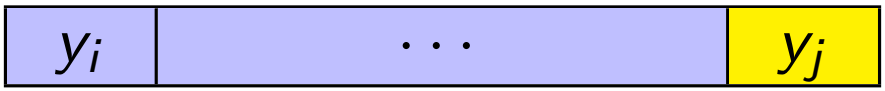

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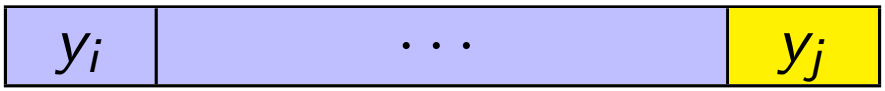

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$$\begin{aligned} P(y_i \text{ and } y_j \text{ are compared}) &= P(Case1) + P(Case3) \\ &= \frac{1}{j-i+1} + \frac{1}{j-i+1} \\ &= \frac{2}{j-i+1} \end{aligned}$$

From Equation (1) and (2),

$$\begin{aligned} E[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\ &= 2\sum_{i=1}^{n-1}(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-i+1}) \\ &< 2\sum_{i=1}^{n-1}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}) \\ &< 2n(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}) \end{aligned}$$

The quantity $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, denoted H_n, is called the n^{th} **Harmonic number** and is in the range $[\ln n, 1 + \ln n]$. Therefore,

$$E[X] < 2n(H_n) \leq 2n \ln n + 2n = \theta(n \ln n)$$

MAX-CUT : Problem Definition

- Input : An undirected graph G(V,E) where V : Set of vertices E: Set of Edges
- Output : A partition of vertices V_1, V_2 such that the no. of edges crossing from V_1 to V_2 is maximised

Easy Example

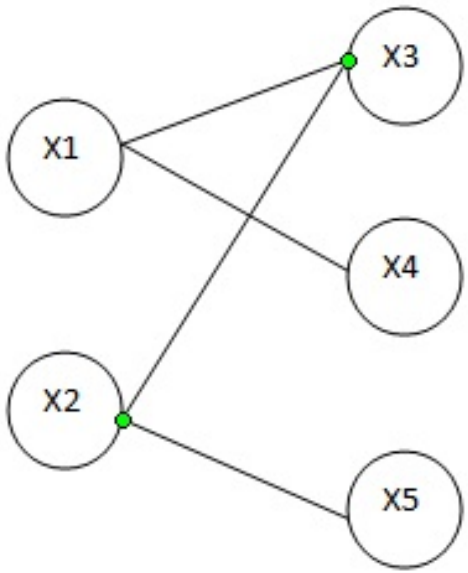

Figure : A Bipartite Graph

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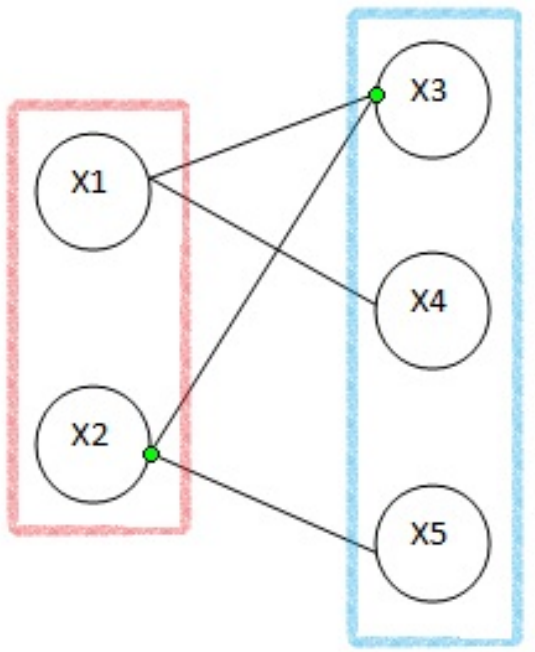

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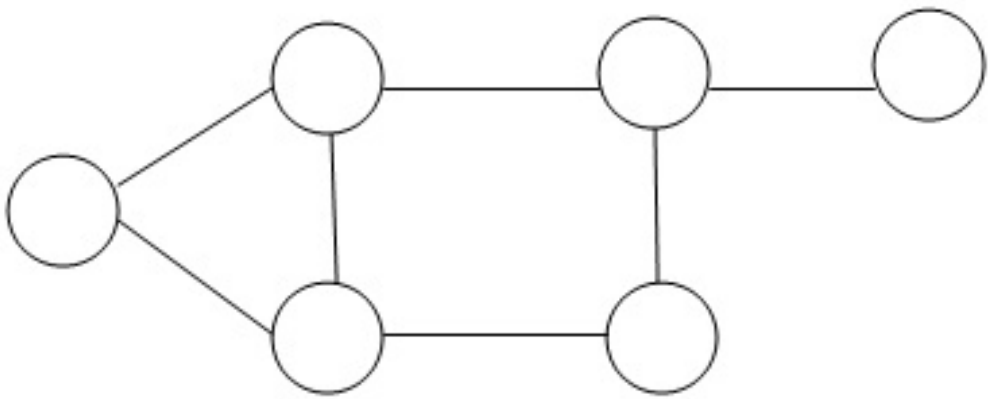

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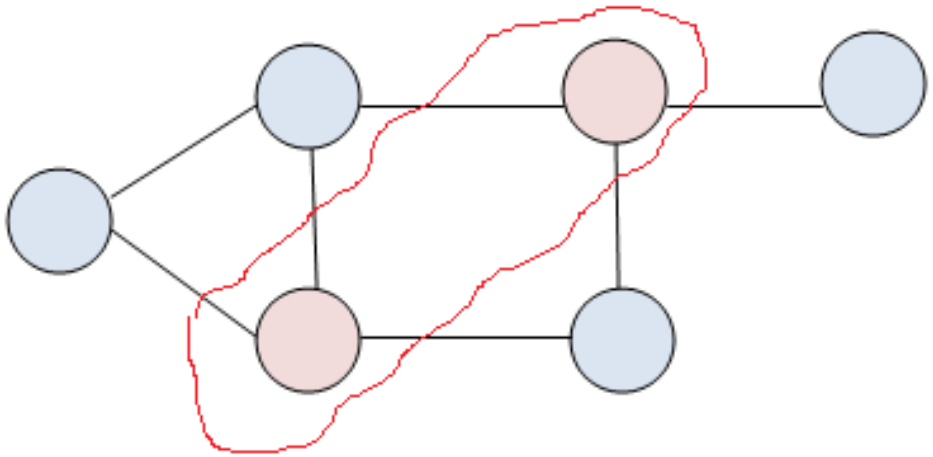

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- The total no. of ways the vertex set can be partitioned is $2^{\|V\|}$
- Brute force would require exponential no. of trials, which is too much hard work!!
- Can we do better ?

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Standard Approaches

- Greedy?
- Divide and Conquer?
- Dynamic Programming?

Randomized Approach Example

- A Simple randomized algorithm that provides (probabilistic) guarantees a cut of size $\|E\|/2$
- Algorithm Initialize the partitions V_1 and V_2 as empty sets
  For each vertex $v \in V$, flip a fair coin. If outcome is heads then put vertex in partition V_1 else put it into partition V_2.

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Pseudo-code

1. **Initialize:** Select a vertex $v \in V$ randomly
2. Remove a vertex $v \in V$ from the vertex set. Toss a "fair" coin. If the outcome is head the assign it to partition V_1 else assign it to vertex V2.
3. Repeat above step till all vertices are assigned a partition .

Randomized Approach Example

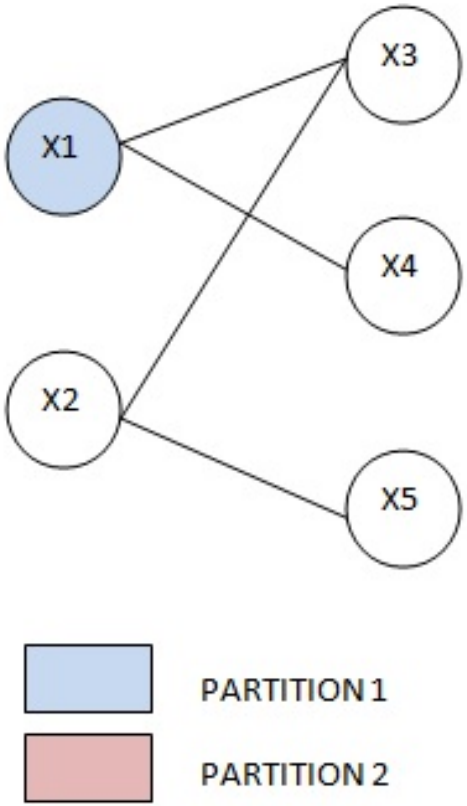
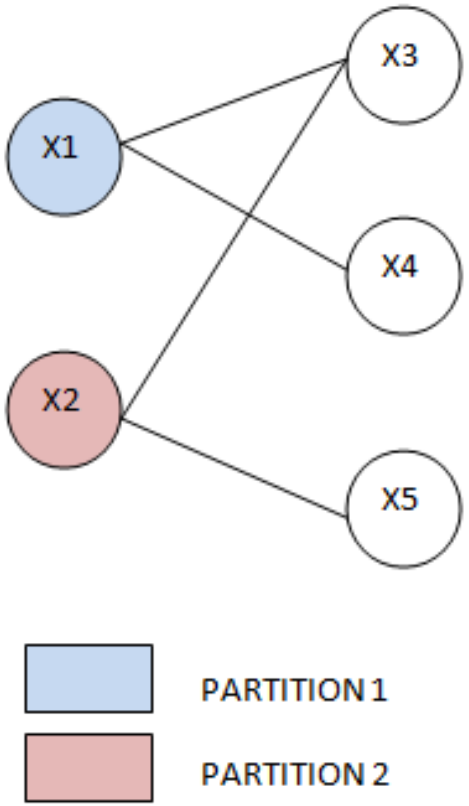


Figure : Coin Toss Sequence = H

Randomized Approach Example



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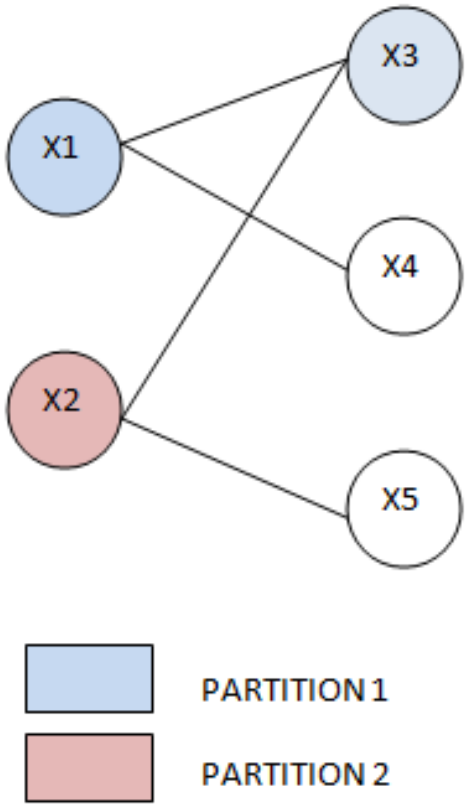

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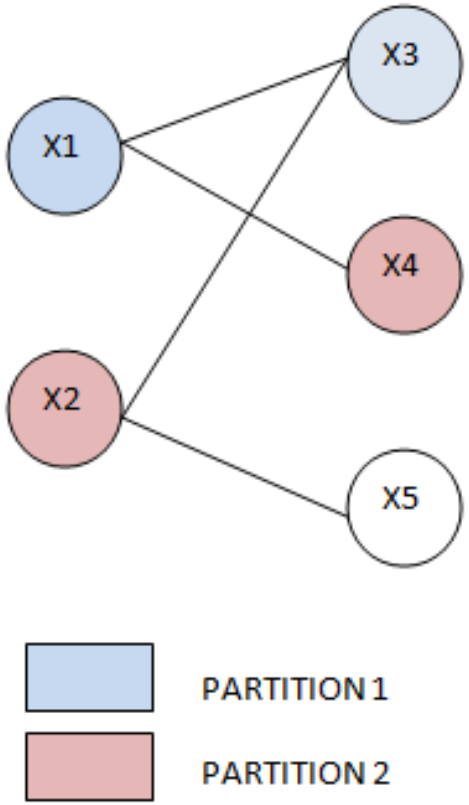

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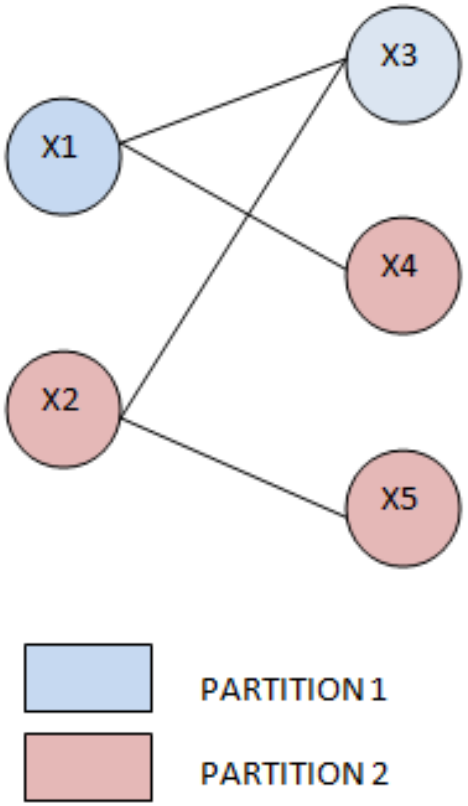

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Does it Work??

Let X_e be a random variable s.t.

$$X_e = \begin{cases} 1 & \text{if Edge } e = (v_i, v_j) \text{ crosses the partition} \\ 0 & \text{otherwise} \end{cases}$$

- Let X be a random variable indicating the no. of edges crossing the cut. Then $X = X_1 + X_2 + \cdots + X_e$
- Expected no. of edges crossing the cut = $E[X] = E[X_1 + X_2 + . + X_e] = \sum_{i=1}^{e} E[X_i]$
- $E[X_e] = 1 \times \Pr(\text{Edge } e \text{ is chosen}) + 0 \times \Pr(\text{Edge } e \text{ is not chosen})$

The 4 cases of partitioning

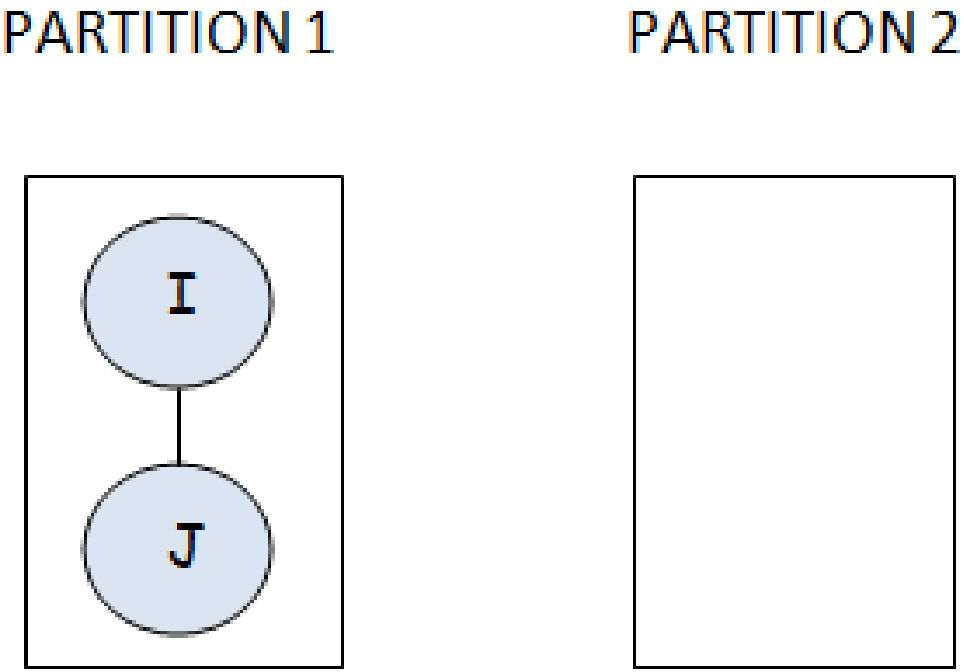

Figure : Case 1 $i \in V_1$ $j \in V_1$

$$\Pr(i \in V_1, j \in V_1) = \Pr(i \in V_1) \times \Pr(j \in V_1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

The 4 cases of partitioning

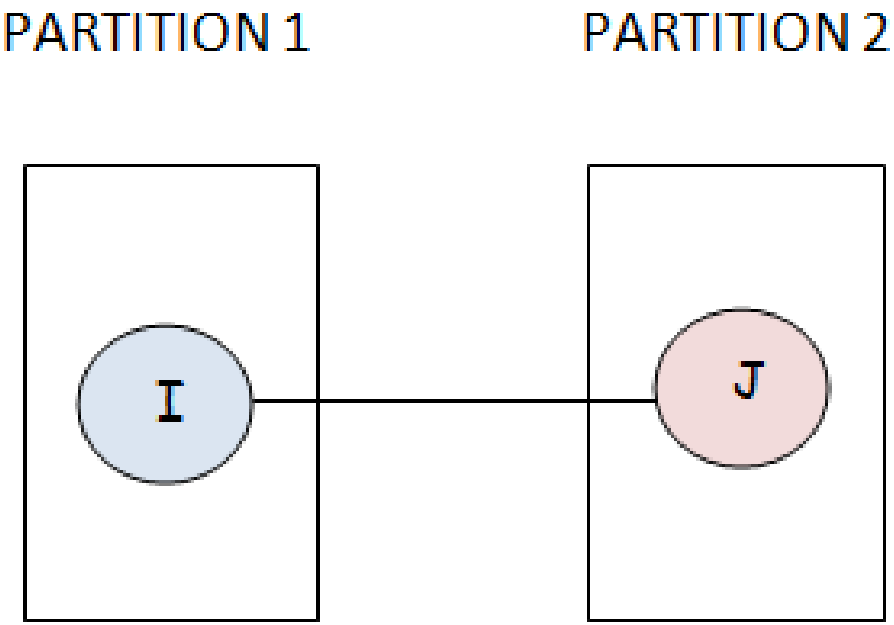

Figure : Case 2 $i \in V_1$ $j \in V_2$

$$\Pr(i \in V_1, j \in V_2) = \Pr(i \in V_1) \times \Pr(j \in V_2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

The 4 cases of partitioning

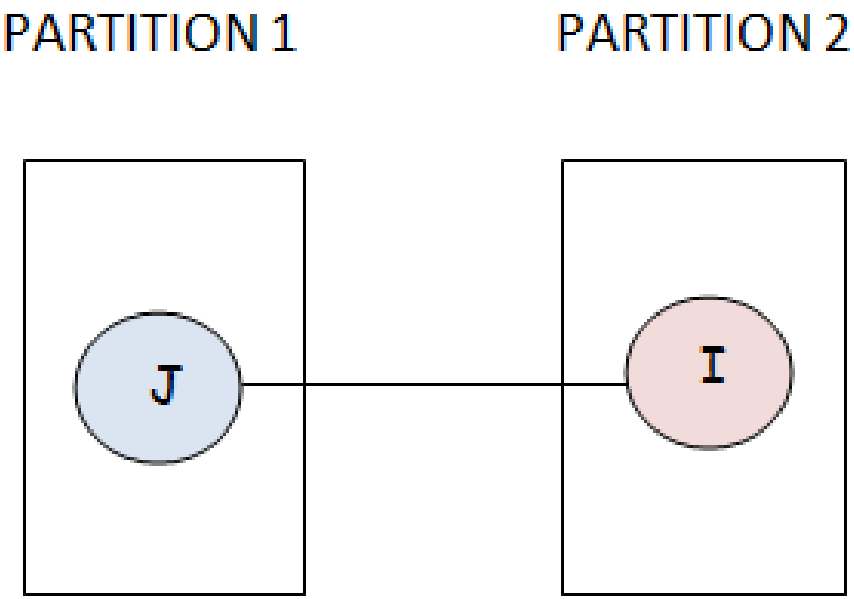

Figure : Case 3 $i \in V_2$ $j \in V_1$

$$\Pr(i \in V_2, j \in V_1) = \Pr(i \in V_2) \times \Pr(j \in V_1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

The 4 cases of partitioning

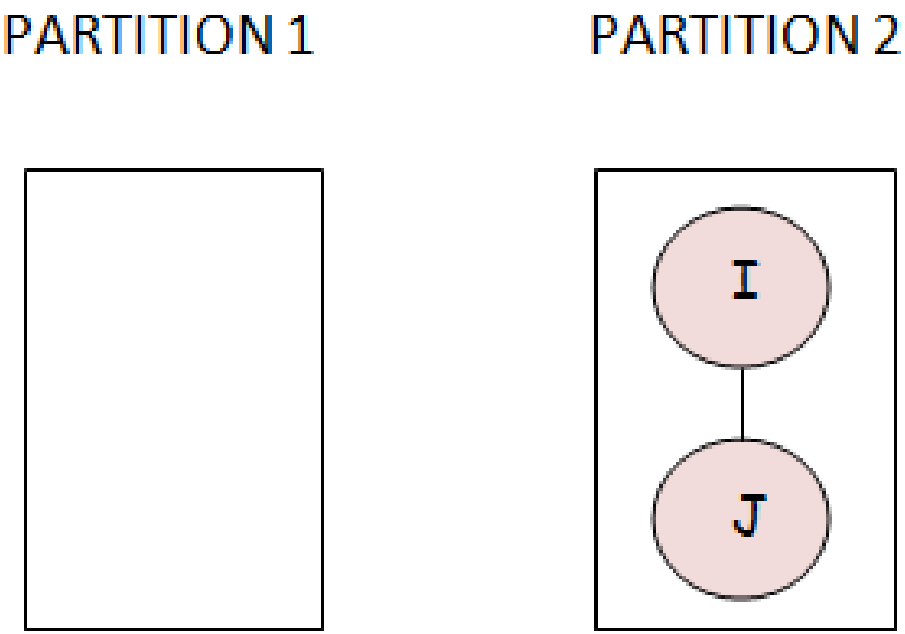

Figure : Case 4 $i \in V_2$ $j \in V_2$

$$\Pr(i \in V_2, j \in V_2) = \Pr(i \in V_2) \times \Pr(j \in V_2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Correctness Contd.

Partition Vertex i	Partition Vertex j	Edge (I,j)	Probability
V_1	V_1	Absent	¼
V_1	V_2	Present	¼
V_2	V_1	Present	¼
V_2	V_2	Absent	¼

P(Edge E is Present) = ¼ + ¼ = ½

- $E[X_{(i,j)}] = 1 \times \Pr(\text{Edge } e \text{ is chosen}) + 0 \times \Pr(\text{Edge } e \text{ is not chosen}) = 1 \times \frac{1}{2} + 0 \times \frac{1}{2} = 1/2$
- Expected no. of edges = $E[X] = E[X_1 + X_2 + . + X_e] = \sum_{i=1}^{e} E[X_i] = \sum_{i=1}^{e} 1/2 = e/2$

Logical and Practical Difficulties with Randomization

- True randomness is hard to replicate, as famously countered by Einstein :

  GOD DOES NOT PLAY DICE

- A large scale randomized operation could need upto billions of randomized bits per second

- Hard to produce that many no. of uncorrelated bits.

Derandomization

- Some Randomized algorithms have deterministic counterparts that provide similar guarantees on output.
- The cost of de-randomization is the added complexity
- Can we derandomize maxcut ??

The Chimp Strategy

- A chimp is given a binary tree in which the leaf nodes are weighed by different no. of bananas
- Objective : Starting from the root node, get to a leaf having atleast $n/2$ bananas
- Incentive?? : Obvious!!

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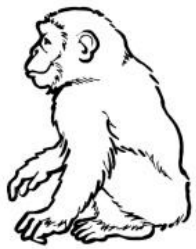
The chimp is given a map of the binary tree but it never takes a look, but always succeeds!!
The **secret** : At each node , the branch with the more no. of bananas is heavier and hence a little less Steep



Back to MAX-CUT

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Correctness

- Proof by induction :
- Hypothesis: At any iteration i, $E[X|v1, v2, , vi] \geq m/2$
- Base case : $E[X] \geq m/2$ @ iteration 0
- Induction step: Assuming $E[X|v1, v2, , vi - 1] \geq m/2$, where $v1 \ldots v_{i-1}$ are as partitioned by the algorithm

More About Randomization

- Is randomization more powerful than determinism ??
- Are there upper bounds on increase in complexity on derandomization ??

Thank you!